SUFE: Algorithmic Game Theory Summer School	Problem Set 1a Solutions
Posted at: July 23, 2017	*Solutions are contributed by Eric Balkanski.*

Solutions to problems in the problems set 1a.

1. **Path Zero-Sum Games.** Let P_1, P_2, and P_3 be players 1, 2, and 3 respectively. We denote their strategies by $\mathbf{x}, \mathbf{y}_1$, and $\mathbf{y}_2$. The remaining of the notation is similar as in lecture. The LP for P_1 if he is forced to announce his strategy in advance is

$$\begin{aligned} & \max z_2 + z_3 \\ \text{s.t.} \quad & \mathbf{x}^\intercal A^{1,2} \geq z_2 \mathbf{1}^\intercal \qquad\qquad LP(1,p) \\ & \mathbf{x}^\intercal A^{1,3} \geq z_3 \mathbf{1}^\intercal \\ & \mathbf{x}^\intercal \mathbf{1} = 1, \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

Notice that the maximum of $z_2 + z_3$ is the same as $\max_{\mathbf{x}} \min_{\mathbf{y}_2, \mathbf{y}_3} \mathbf{x}^\intercal (A^{1,2}\mathbf{y}_2 + A^{1,3}\mathbf{y}_3)$. The dual of $LP(1)$ can be written as

$$\begin{aligned} & \max z_1 \\ \text{s.t.} \quad & A^{2,1}\mathbf{y}_2 + A^{3,1}\mathbf{y}_3 \geq z_1 \mathbf{1} \qquad\qquad LP(1,d) \\ & \mathbf{y}_2^\intercal \mathbf{1} = \mathbf{y}_3^\intercal \mathbf{1} = 1, \mathbf{y}_2, \mathbf{y}_3 \geq \mathbf{0} \end{aligned}$$

Assume $(\mathbf{x}, z_2, z_3)$ and $(\mathbf{y}_2, \mathbf{y}_3, z_1)$ are optimal solutions to $LP(1,p)$ and $LP(1,d)$, then we claim that $(\mathbf{x}, \mathbf{y}_2, \mathbf{y}_3)$ is a Nash equilibrium for this game.

We show that each player is best responding against the other players' strategies. We begin with P_1. Since these solutions are feasible, we have

$$\begin{aligned} \mathbf{x}^\intercal A^{1,2} \geq z_2 \mathbf{1}^\intercal &\Rightarrow \mathbf{x}^\intercal A^{1,2}\mathbf{y}_2 \geq z_2 && (1) \\ \mathbf{x}^\intercal A^{1,3} \geq z_3 \mathbf{1}^\intercal &\Rightarrow \mathbf{x}^\intercal A^{1,3}\mathbf{y}_3 \geq z_3 && (2) \\ A^{2,1}\mathbf{y}_2 + A^{3,1}\mathbf{y}_3 \geq z_1 \mathbf{1} &\Rightarrow \mathbf{x}'^\intercal (A^{2,1}\mathbf{y}_2 + A^{3,1}\mathbf{y}_3) \geq z_1, \forall \mathbf{x}' \\ &\Rightarrow \mathbf{x}'^\intercal (A^{1,2}\mathbf{y}_2 + A^{1,3}\mathbf{y}_3) \leq -z_1, \forall \mathbf{x}' && (3) \end{aligned}$$

If P_2 and P_3 play $\mathbf{y}_2, \mathbf{y}_3$, by (3), P_1 gets at most $-z_1 = z_2 + z_3$ by strong duality. P_1 gets $z_2 + z_3$ by playing $\mathbf{x}$ by (1) and (2). Thus $\mathbf{x}$ is best response to $\mathbf{y}_2, \mathbf{y}_3$

Next, for P_2 and P_3, we have

$$\begin{aligned} A^{2,1}\mathbf{y}_2 + A^{3,1}\mathbf{y}_3 \geq z_1 \mathbf{1} &\Rightarrow \mathbf{x}^\intercal A^{2,1}\mathbf{y}_2 + \mathbf{x}^\intercal A^{3,1}\mathbf{y}_3 \geq z_1 && (1) \\ \mathbf{x}^\intercal A^{1,2} \geq z_2 \mathbf{1}^\intercal &\Rightarrow \mathbf{x}^\intercal A^{1,2}\mathbf{y}_2' \geq z_2, \forall \mathbf{y}_2' \\ &\Rightarrow \mathbf{x}^\intercal A^{2,1}\mathbf{y}_2' \leq -z_2, \forall \mathbf{y}_2' && (2) \\ \mathbf{x}^\intercal A^{1,3} \geq z_3 \mathbf{1}^\intercal &\Rightarrow \mathbf{x}^\intercal A^{1,3}\mathbf{y}_3' \geq z_3, \forall \mathbf{y}_3' \\ &\Rightarrow \mathbf{x}^\intercal A^{3,1}\mathbf{y}_3' \leq -z_3, \forall \mathbf{y}_3' && (3) \end{aligned}$$

If P_1 plays $\mathbf{x}$, by (2) and (3), P_2 and P_3 get at most $-z_2$ and $-z_3$ respectively. P_2 and P_3 get a combined value of $z_1 = -z_2 - z_3$ by playing $\mathbf{y}_2$ and $\mathbf{y}_3$ by (1) and strong duality. Since they get combined value of $-z_2 - z_3$ and that individually they can get at most $-z_2$ and $-z_3$, it must be the case that P_2 and P_3 get $-z_2$ and $-z_3$ respectively and that this is best responding.

2. **Nash Equilibrium Computation.**

(a) Fix S and T. Consider the following linear program where x and y are the variables:

$$
\begin{aligned}
e_i^\intercal R y &\geq e_j^\intercal R y && \text{for all } i \in S, j \in [n] \\
x^\intercal C e_i &\geq x^\intercal C e_j && \text{for all } i \in T, j \in [m] \\
\sum_{i \in S} x_i &= 1 \\
\sum_{i \in T} y_i &= 1 \\
x_i &= 0 && \text{for all } i \notin S \\
y_i &= 0 && \text{for all } i \notin T \\
x_i &> 0 && \text{for all } i \in S \\
y_i &> 0 && \text{for all } i \in T
\end{aligned}
$$

First, assume that a Nash equilibrium (x, y) with supports S and T exists, so $x_i > 0$ for $i \in S$ and $y_i > 0$ for $i \in T$. Then, by the forward direction of Definition 7 from lecture 5, (x, y) is a solution to the above LP. Next, assume (x, y) is a solution to this LP, then by the reverse direction of this same definition 7 from lecture 5, it is a Nash equilibrium. In addition, it is clear that it is supported on S and T. Thus, the algorithm which solves the above LP computes a Nash equilibrium supported on S and T if a Nash with supports S and T exists.

(b) Consider algorithm $\mathcal{A}$ which loops over all pairs (S, T) such that $S \subseteq [n]$ and $T \subseteq [m]$ and for each such pair, solves the above linear program. Let (x, y) be a solution to the above LP for the first pair S, T for which the LP has a solution and return (x, y).

We know that every such (R, C) game has a Nash equilibrium. Thus, there exists at least one pair (S, T) such that a Nash equilibrium with supports S and T exists. So $\mathcal{A}$ will find a pair (S, T) such that the LP has a solution and will return some (x, y) by part (a). Also, by part (a), we know that any (x, y) that is a solution to the LP for some (S, T) is a Nash equilibrium. Thus $\mathcal{A}$ computes a Nash equilibrium of the game.

Since the entries of R and C are rational numbers of bit complexity b and since the LP has a number of constraints that is linear in n amd m , the LP can be solved in $O(\text{poly}(n, m, b))$ time. There are 2^{n+m} pairs (S, T) such that $S \subseteq [n]$ and $T \subseteq [m]$. Thus $\mathcal{A}$ has total runtime $O(2^{n+m} \cdot \text{poly}(n, m, b))$.

(c) We set up an LP similar to before, but with the approximate Nash equilibrium constraints as stated in the problem, and optimize in order to minimize ϵ. We know that there is a solution with accuracy at most $\hat{\epsilon}$, but since the LP will only find a solution with polynomial bit complexity, it must output the solution with accuracy $\epsilon = 0$.

3. **Player Exchangeability.**

At a high level, we use a similar technique as Nash's proof but use a different function f with which we use Brouwer's fixed point theorem. We denote $S := S_1 = S_2$.

We keep the same $Gain_{p;s_p}(x)$ function as in lecture for $p > 2$. Define x' to be the vector x such that the strategies of the first two players are the average of their strategies according to x, i.e., $x'_1 = x'_2 = (x_1 + x_2)/2$ and $x'_i = x_i$ for $i > 2$. We then define:

$$
\begin{aligned}
Gain_{1;s}(x) &= \max\{u_1(s, x'_{-1}) - u_1(x'), 0\} \\
Gain_{2;s}(x) &= \max\{u_2(s, x'_{-2}) - u_2(x'), 0\}
\end{aligned}
$$

Note that by exchangeability, $u_1(s, x'_{-1}) = u_2(s, x'_{-2})$ and $u_1(x') = u_2(x')$ since $x'_1 = x'_2$. Thus for all $s \in S$,

$$Gain_{1;s}(x) = Gain_{2;s}(x)$$

The function f mapping x to y is modified as follows. For $p > 2$, we keep the same mapping as in lecture and define

$$y_1(s) := \frac{x_1(s) + x_2(s) + Gain_{1,s}(x')}{2 + \sum_{s' \in S} Gain_{1;s'}(x')}$$
$$y_2(s) := \frac{x_1(s) + x_2(s) + Gain_{2,s}(x')}{2 + \sum_{s' \in S} Gain_{2;s'}(x')}$$

Since $Gain_{1;s}(x') = Gain_{2;s}(x')$ for all s, we have

$$y_1(s) = y_2(s)$$

The denominator still ensures that $\sum_{s \in S} y(s) = 1$. It is also the case that f is still continuous, and similarly as in lecture, f has a fixed point by Brouwer's theorem.

We claim that a fixed point x of f is such that (1) it is a Nash equilibrium and (2) $x_1 = x_2$. Since $y_1(s) = y_2(s)$, it must be the case that $x_1 = x_2$ for a fixed point x and (2) immediately holds. Next, we show that $Gain_{p;s_p}(x) = 0$ for all $p > 2$ and $Gain_{1;s}(x') = Gain_{2;s}(x') = 0$.

For $p > 2$, $Gain_{p;s_p}(x) = 0$ for all s_p identically as in lecture. Thus $p > 2$ is best responding for a fixed point. For $p = 1$ or $p = 2$, assume by contradiction that there exists fixed point x and some s such that $Gain_{1,s}(x') > 0$. It has to be the case that $x_1(s) + x_2(s) > 0$ otherwise x is not a fixed point. Thus, $x'_1(s) > 0$. Next, note that

$$u_1(x') = u_2(x') = \sum_{s' \in S} x'_1(s') u_1(s, x'_{-1}) = \sum_{s' \in S} x'_2(s') u_2(s', x'_{-2})$$

Since $x'_1(s) > 0$ and $u_1(s, x'_{-1}) > u_1(x')$, there must exist some s'' such that $x_1(s'') > 0$ and $u_1(s'', x'_{-1}) - u_1(x') < 0$. Thus, this strategy satisfies $Gain_{1;s''}(x) = 0$. So

$$y_1(s'') = \frac{x_1(s'') + x_2(s'') + Gain_{1,s''}(x) + Gain_{2,s''}(x)}{2 + \sum_{s'_1 \in S_1} Gain_{1;s'_1}(x)} < (x_1(s'') + x_2(s''))/2$$

since the denominator is strictly greater than 2. Since it must be the case that $x_1(s'') = x_2(s'')$ for a fixed point, we have $y_1(s'') < (x_1(s'')$ which is a contradiction with x being a fixed point.

Thus $Gain_{1;s}(x) = Gain_{2;s}(x) = 0$ for all s for a fixed point x. Since $x_1 = x_2$,

$$0 = Gain_{1;s}(x') = Gain_{2;s}(x') = \max\{u_1(s, x_{-1}) - u_1(x), 0\} = \max\{u_1(s, x_{-1}) - u_1(x), 0\}$$

and we obtain that the first two players are also best responsive. We conclude that we obtained a Nash Equilibrium with $x_1 = x_2$ as desired.

4. **Splitting the Rent.**

We consider a grid where a vertex (x, y), $0 \leq x, y \leq 3000$, corresponds to $p_R = x$, $p_G = y$, and $p_B = 3000 - x - y$.

We then describe a coloring of this grid. If $x + y > 3000$, color (x, y) with B (blue). Otherwise, ask C,V, and M which room they prefer at prices $p_R = x$, $p_G = y$, and $p_B = 3000 - x - y$. If V and M prefer the same room, color (x, y) with the name of that room. Otherwise, color it with the name of the favorite room for C.

We first verify that the boundaries of this grid have the standard coloring necessary for Sperner's lemma. If $x = 0$ and $0 < y < 3000$, then all three players prefer room R by the assumption of the question and the coloring of (x, y) is R. Similarly, if $y = 0$ and $0 < x < 3000$, then all three players prefer room G and the coloring is G. If $x + y > 3000$, then the coloring is B by definition. Finally, $(0, 0)$ is either colored R or G, $(0, 3000)$ either R or B, and $(3000, 0)$ either G and B by the assumption. Thus, the boundaries have a standard coloring.

Next, by Sperner, there is a tri-chromatic triangle in this grid. A vertex is either colored due to C or to VM (if V and M have the same preference at that vertex). There are four cases for how many of the three vertices of this triangle got colored due to C, the different colors for each case are without loss of generality. We show that in each case, we can assign to a player his favorite room for the prices at one of the three vertices of the tri-chromatic triangle. Picking that assignment with the prices for the top-right corner, we then obtain that everyone is almost happy since the vertices of a same triangle have coordinate-wise distance at most 1.

- R, G, B **due to** C**, none due to** VM**.** V and M have different favorite rooms at the vertex colored R. We assign each of them to their favorite room at that vertex. We assign C to the remaining room.
- R, G **due to** C**,** B **due to** VM**.** V and M have different favorite rooms at the vertex colored R. At least one of them has to not be B, we assign that room to whoever of V or M preferred it. We assign B to the other one of V or M. We assign the last room to C.
- R **due to** C**,** G, B **due to** VM**.** One of V and M has to prefer a room that is not R at the vertex colored R. We assign that room that is not R to that player. We assign room R to C and the remaining room to the remaining player.
- **none due to** C**,** R, G, B **due to** VM**.** We assign C his favorite room at any vertex of the triangle. We assign the two remaining rooms to V or M arbitrarily.