SUFE: Algorithmic Game Theory Summer School	Problem Set 2b
Due: July 28, 2017, 23:59pm	*Submit solutions to as many problems as you can.*

Please send your solutions to **nikolai@mail.shufe.edu.cn**. You may submit typed in latex pdfs, or scans/photos of your handwritten solutions.

- **Advertising on a search engine.** Every time that you search for a keyword on a search engine an auction takes place to allocate the advertising slots on the webpage. A stylized model of the auction can be described as follows: a set of n slots is auctioned to a set of n advertisers, henceforth bidders. We follow the convention that slot 1 is the slot at the top of the page and slot n the bottom and we will denote with $[n] = \{1, \ldots, n\}$. Assuming that there are as many slots as advertisers is without loss of generality.

  Stylized sponsored search auction model. Each bidder $i \in [n]$ receives a value v_i if the person visiting the webpage clicks on their ad. This is a proxy for the expected revenue that the advertiser expects to receive from a visitor to his site. When the ad of bidder $i \in [n]$ appears in slot $j \in [n]$ then it has some probability of being clicked. For simplicity we will assume that this probability is only a function of the slot j and not of the bidder and we will denote it with α_j and refer to it as click-through-rate. As is natural, higher slots on the page are more visible, hence we will assume that $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$ and denote with $\alpha = (\alpha_1, \ldots, \alpha_n)$.

  The utility of a bidder i when his ad appears in slot j and he is asked to pay an amount of money p is quasi-linear, i.e.:

  $$U_i(j, p) = \alpha_j v_i - p \tag{1}$$

  where $\alpha_j v_i$ is the total expected value the bidder receives from appearing in slot j. The quantities α are typically estimated by the search engine using historical data and machine learning algorithms. For simplicity, we will assume that these quantities are known by the search engine.

  A feasible allocation in this setting is simply a matching which assigns a bidder i to a slot $\sigma(i)$ and a slot j to a bidder $\pi(j) = \sigma^{-1}(j)$. The social welfare of any such allocation is:

  $$SW(b) = \sum_{i=1}^{n} \alpha_{\sigma(i)} v_i \tag{2}$$

  Denote with σ^*, the matching that maximizes the social welfare and with OPT the optimal social welfare:

  $$OPT = \sum_{i=1}^{n} \alpha_{\sigma^*(i)} v_i \tag{3}$$

  Generalized Second Price (GSP) auction. A stylized version of the auction that is being run by most major search engines is what is known as the generalized second price auction.

  - Each bidder submits a bid b_i which is interpreted as his willingness to pay for each click he receives, i.e. pay-per-click bid. Let $b = (b_1, \ldots, b_n)$.
  - Bidders are ordered in decreasing order of bids (breaking ties lexicographically). Denote with $\sigma(i, b)$ the slot that bidder i receives and with $\pi(j, b)$ the bidder allocated in slot j under this ordering.
  - Each bidder pays only when he is clicked and he pays the next highest bid, i.e. a player in slot j pays $b_{\pi(j+1)}$ per-click.

Assume that all bidders know each others valuations and we will denote with $v = (v_1, \ldots, v_n)$ the valuation profile. Hence, the Generalized Second Price Auction defines a complete information game among the bidders.

1. If all players submit their true valuation as their bid, i.e. $b_i = v_i$, then is the allocation of GSP the welfare maximizing one?
2. Write down the utility of each player as a function of the bid vector b and the click-through-rates α, i.e. $u_i(b; \alpha)$. Then write the pure Nash equilibrium constraints.
3. Suppose that there is effectively only one slot that is auctioned, i.e. $\alpha_1 = 1$ and $\alpha_2 = \ldots = \alpha_n = 0$. Then is bidding truthfully an equilibrium of GSP?
4. Suppose that there are two slots and two bidders, with $\alpha_1 = 1$, $\alpha_2 = 0.5$ and $v_1 = 1$ $v_2 = 0.8$. Is bidding truthfully a Nash equilibrium? Why or why not? Are there efficient pure Nash equilibria (i.e. pure Nash equilibria that maximize the social welfare)? Are there inefficient pure Nash equilibria and what is the ratio of the social welfare at these equilibria vs the optimal social welfare?
5. Show that if a player bids half of his value, i.e. $v_i/2$, then for any bid profile bid profile b:

$$u_i\left(\frac{v_i}{2}, b_{-i}; \alpha\right) + \alpha_{\sigma^*(i)} b_{\pi(\sigma^*(i))} \geq \frac{1}{2}\alpha_{\sigma^*(i)} v_i \tag{4}$$

   i.e. either the player's utility is a constant fraction of the player's contribution to the optimal welfare, or the bid of the player who is allocated at the optimal slot for player i is high.

6. Assuming that no-player bids above his value, i.e. $b_i \leq v_i$, show that the GSP auction is $(1/2, 1)$-smooth, i.e. show that for any player i and for any valuation profile there exists a bid b_i^*, such that for any bid profile b:

$$\sum_{i=1}^{n} u_i(b_i^*, b_{-i}; \alpha) \geq \frac{1}{2} OPT - SW(b) \tag{5}$$

   Conclude that the average welfare $\frac{1}{T}\sum_{t=1}^{T} SW(b^t)$ of no-regret dynamics for T iterations in this game has social welfare at least $\frac{1}{4} OPT - n \cdot \epsilon(T)$, when each player uses an online learning algorithm with regret $\epsilon(T)$.