

## Lecture 1.2: Nash's Theorem

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We formally define games and some solution concepts. We then proceed to prove Nash's theorem on the existence of Nash equilibrium in every finite game using Brouwer's fixed point theorem.

## 1 Games

We start with a formal definition of games, as relevant for the *complete information setting*.

**Definition 1.** A normal-form game is specified by:

- the number of players  $n$ ; we denote the set of players by  $[n] = \{1, 2, \dots, n\}$ ;
- for each player  $p \in [n]$ :
  - a finite set of pure strategies, or actions,  $S_p$  available to player  $p$ ;
  - a utility function  $u_p : \prod_{p \in [n]} S_p \rightarrow \mathbb{R}$ , specifying the payoff to player  $p$  for each selection of pure strategies by the players of the game.

We often summarize this information in a tuple  $\langle n, (S_p)_{p \in [n]}, (u_p)_{p \in [n]} \rangle$ .

Under complete information, it is assumed that  $\langle n, (S_p)_{p \in [n]}, (u_p)_{p \in [n]} \rangle$  are known to all players.

Later in this mini-course (in applications to mechanism design), we will study games in the *incomplete information setting*, where players may not know the utility functions of the other players. We postpone further discussion.

**Notation.** Relative to a game specification, we introduce some useful concepts and notation:

**Definition 2.** Let  $\langle n, (S_p)_{p \in [n]}, (u_p)_{p \in [n]} \rangle$  be a normal-form game. Then

- the set of mixed strategies available to player  $p$  are all distributions over  $S_p$ , denoted
- $$\Delta^{S_p} = \left\{ x_p \in \mathbb{R}_{\geq 0}^{S_p} \mid \sum_{s_p \in S_p} x_p(s_p) = 1 \right\};$$
- an element of  $S := \prod_{p \in [n]} S_p$  is called a pure strategy profile;
  - an element of  $\Delta := \prod_{p \in [n]} \Delta^{S_p}$  is called a mixed strategy profile;
  - if  $s \in S$ , we denote by  $s_p$  the pure strategy of player  $p$  in  $s$ ; in particular,  $s_p \in S_p$ ; we also denote by  $s_{-p}$  the vector of pure strategies of all players except  $p$  in  $s$ ; in particular,  $s_{-p} \in \prod_{q \neq p} S_q$ ;
  - similarly, if  $x \in \Delta$ , we denote by  $x_p$  the mixed strategy of player  $p$  in  $x$ ; in particular,  $x_p \in \Delta^{S_p}$ ; we also denote by  $x_{-p}$  the vector of mixed strategies of all players except  $p$  in  $x$ ; in particular,  $x_{-p} \in \prod_{q \neq p} \Delta^{S_q}$ ;
  - finally, we often refer to pure strategies as “actions”, reserving the term “strategy” for referring to a mixed strategy.

When players use randomized strategies it is assumed that they sample from their mixed strategies *independently of the other players*. Hence, given a mixed strategy profile  $x \in \Delta$ , the expected payoff of player  $p$  is given by

$$u_p(x) = \sum_{s \in S} u_p(s) \prod_{q \in [n]} x_q(s_q),$$

where, for a pure strategy profile  $s \in S$ ,  $\prod_{q \in [n]} x_q(s_q)$  is just the probability that  $s$  is sampled, when players independently sample their mixed strategies. We use the following shorthand for the above expression:

$$u_p(x) = \mathbb{E}_{s \sim x} [u_p(s)],$$

where it is implied in the notation “ $s \sim x$ ”, that  $s \in S$  is drawn by having each player  $q \in [n]$  independently draw a sample from his mixed strategy  $x_q$ .

## 2 Solution Concepts

Solution concepts aim to capture how rational players should play a game. We discuss a few of them here.

### 2.1 Dominance

We start with the notion of dominance, and the concept of dominant strategy equilibrium.

**Definition 3** (Dominance). *We say that a strategy  $x_p \in \Delta^{S_p}$  very weakly dominates another strategy  $x'_p \in \Delta^{S_p}$  iff for all  $x_{-p}$ :*

$$u_p(x_p; x_{-p}) \geq u_p(x'_p; x_{-p}).$$

*If additionally the inequality is strict for at least one  $x_{-p}$ , then we say that  $x_p$  weakly dominates  $x'_p$ . Finally, if the inequality is strict for all  $x_{-p}$ , we say that  $x_p$  strictly dominates  $x'_p$ .*

**Definition 4** (Dominant strategy equilibrium). *A mixed strategy profile  $x$  is a strictly (respectively weakly or very weakly) dominant strategy equilibrium iff for each player  $p$ ,  $x_p$  strictly (respectively weakly or very weakly) dominates every other  $x'_p$ .*

For example, in Prisoner’s dilemma, strategy ‘betray’ strictly dominates strategy ‘cooperate’ as well as any strict mixture of ‘betray’ and ‘cooperate’. Hence, the strategy profile (‘betray’, ‘betray’) is a strictly dominant strategy equilibrium. Clearly, a strictly/weakly dominant strategy equilibrium must be unique, but there could be multiple very weakly dominant strategy equilibria in a game.

### 2.2 Removal of Dominated Strategies (not covered in class-extra reading)

If an action  $s_p$  of player  $p$  is strictly dominated by some mixed strategy  $x_p$  (w.l.o.g.  $x_p$  does not have  $s_p$  in its support), we may postulate that  $p$ —being rational—will never play this action, as it can never be a best response or in the support of a best response to the other players’ strategies. Now, assuming that some other player  $q$  is also rational and believes that player  $p$  is rational, we may postulate that  $q$  believes that  $p$  will never use action  $s_p$ . This may make some action  $s_q$  of player  $q$  become strictly dominated by some mixed strategy  $x_q$ , even if it was not dominated before the elimination of  $s_p$ . We can continue this process iteratively eliminating actions from players’ strategy sets until no further elimination is possible. This motivates the following definition.

**Definition 5** (Strategies Surviving Elimination of Dominated Strategies). *A mixed strategy profile is said to survive the iterated elimination of strictly dominated strategies if every action in the support of every player’s strategy cannot be removed through any sequence of iterated elimination of strictly dominated strategies.*

We can simplify our definition by establishing the following property.

**Exercise 1.** *The actions surviving the iterated elimination of strictly dominated strategies are not dependent on the exact sequence of elimination.*

In the game “guess two-thirds of the average,”, the all-0 strategy profile is the unique profile surviving the iterated elimination of strictly dominated strategies.

**Exercise 2.** *For a normal form game specified explicitly (i.e. by specifying each player’s actions and each player’s utility for every pure strategy profile), show that there is a polynomial-time algorithm for checking whether there exist  $p, s_p \in S_p, x_p \in \Delta^{S_p}$  such that  $s_p$  is strictly dominated by  $x_p$ . Likewise for weak dominance.*

### 2.3 Nash Equilibrium

One of the most influential concepts in Game Theory is the concept of Nash equilibrium defined next.

**Definition 6** (Nash Equilibrium). *A mixed strategy profile  $x \in \Delta$  is a Nash equilibrium iff for all  $p \in [n]$  and  $x'_p \in \Delta^{S_p}$ :*

$$u_p(x) \geq u_p(x'_p; x_{-p}).$$

In words,  $x$  is a Nash equilibrium iff no player can strictly increase his or her payoff by switching to a different mixed strategy, if the other players don’t change their strategies. Notice that the expected payoff of a player is a linear function of his own mixed strategy, since

$$u_p(x) \equiv \sum_{s_p \in S_p} x_p(s_p) \cdot u_p(s_p; x_{-p}).$$

Hence, an equivalent definition of Nash equilibrium is the following:

**Definition 7.** *A mixed strategy profile  $x \in \Delta$  is a Nash equilibrium iff for all  $p \in [n]$  and  $s_p, s'_p \in S_p$  such that  $x_p(s_p) > 0$ , we have*

$$u_p(s_p; x_{-p}) \geq u_p(s'_p; x_{-p}).$$

We conclude by showing that the set of Nash equilibria is a restriction of Definition 5.

**Exercise 3.** *Nash equilibria survive the iterated elimination of strictly dominated strategies.*

**Approximate Nash equilibria.** Sometimes we need to relax the Nash equilibrium conditions, allowing for a small margin of improving one’s payoff. This gives rise to notions of approximate equilibrium:

**Definition 8** ( $\epsilon$ -approximate Nash equilibrium). *A mixed strategy profile  $x \in \Delta$  is a  $\epsilon$ -approximate Nash equilibrium iff for all  $p \in [n]$  and  $x'_p \in \Delta^{S_p}$  we have*

$$u_p(x) \geq u_p(x'_p; x_{-p}) - \epsilon.$$

**Definition 9** ( $\epsilon$ -well-supported Nash equilibrium). *A mixed strategy profile  $x \in \Delta$  is a  $\epsilon$ -well-supported Nash equilibrium iff  $\forall p \in [n], s_p, s'_p \in S_p$  such that  $x_p(s_p) > 0$ , we have*

$$u_p(s_p; x_{-p}) \geq u_p(s'_p; x_{-p}) - \epsilon.$$

Notice that these two notions of approximate equilibrium are no longer equivalent. It is easy to see that an  $\epsilon$ -well-supported Nash equilibrium is also an  $\epsilon$ -approximate Nash equilibrium. However, the opposite is not always true.

### 3 Nash's Theorem

One of the most influential results in Game Theory and all of Economics is the following theorem by John Nash, establishing the existence of Nash equilibrium in every finite game. It was shown in a one-page paper published by Nash in 1950 [3], which generalizes von Neumann's earlier result that a Nash equilibrium exists in two-player zero-sum games [5].

**Theorem 1** (Nash [3]). *Every game with a finite number of players and a finite number of actions available to each player has a Nash equilibrium.*

Nash's original proof of this theorem used Kakutani's fixed point theorem for correspondences [2]. A year later Nash simplified [4] his proof to only use Brouwer's fixed point theorem [1]. We present this simplified proof here. Before doing that, we need Brouwer's fixed point theorem. This theorem was proven in the slides using Sperner's lemma.

**Theorem 2** (Brouwer). *Let  $D$  be a convex, compact subset of the Euclidean space. If  $f : D \rightarrow D$  is continuous, then there exists  $x \in D$  such that  $f(x) = x$ .*

The idea behind Nash's proof is to construct a function  $f : \Delta \rightarrow \Delta$  that satisfies the conditions of Brouwer's fixed point theorem such that the fixed point  $x$  is a Nash equilibrium. To do so, we introduce the idea of a gain function.

**Definition 10.** Suppose  $x \in \Delta$  is given. For a player  $p$  and strategy  $s_p \in S_p$ , we define the gain as

$$\text{Gain}_{p;s_p}(x) = \max\{u_p(s_p; x_{-p}) - u_p(x), 0\}.$$

In other words, the gain is equal to the increase in payoff for player  $p$  if he were to switch to pure strategy  $s_p$ , unless the increase is negative in which case the gain is taken to equal 0.

**Proof of Theorem 1:** We define a function  $f : \Delta \rightarrow \Delta$  as follows. For all  $x \in \Delta$ ,  $x \stackrel{f}{\mapsto} y$  where for all  $p \in [n]$  and  $s_p \in S_p$ :

$$y_p(s_p) := \frac{x_p(s_p) + \text{Gain}_{p;s_p}(x)}{1 + \sum_{s'_p \in S_p} \text{Gain}_{p;s'_p}(x)}.$$

In words, function  $f$  tries to boost the probability mass that player  $p$  places on various pure strategies depending on the gains in payoff the player would get by switching to these strategies. The denominator just ensures that  $\sum_{s_p \in S_p} y_p(s_p) = 1$ .

It is easy to see that  $f$  is continuous. Moreover,  $\Delta$  is a product of simplices, so is convex. At the same time,  $\Delta$  is both closed and bounded, so it is also compact. Hence, Brouwer's fixed point theorem ensures the existence of a fixed point of  $f$ .

We claim that any fixed point of  $f$  is a Nash equilibrium. To establish this, it suffices to prove that a fixed point  $x = f(x)$  satisfies:

$$\text{Gain}_{p;s_p}(x) = 0, \quad \forall p \in [n], s_p \in S_p.$$

We proceed by contradiction. Assume that there is some player  $p$  who can improve his payoff by switching to pure strategy  $s_p$ , i.e.

$$\text{Gain}_{p;s_p}(x) > 0.$$

First, it is easy to see that  $x_p(s_p) > 0$ , otherwise  $x$  cannot be a fixed point. Indeed,  $x_p(s_p)$  would be 0, while  $y_p(s_p)$  would be positive.

Given this, we argue next that there must exist some other pure strategy  $s''_p$  such that

$$x_p(s''_p) > 0 \tag{1}$$

and

$$u_p(s''_p; x_{-p}) - u_p(x) < 0. \tag{2}$$

Indeed, notice that

$$u_p(x) \equiv \sum_{s'_p \in S_p} x_p(s'_p) \cdot u_p(s'_p; x_{-p}).$$

Hence, because  $x_p(s_p) > 0$  and  $u_p(s_p; x_{-p}) > u_p(x)$ , there must exist some  $s''_p$  satisfying (1) and (2).

Now notice that a pure strategy  $s''_p$  satisfying (1) and (2) also satisfies  $\text{Gain}_{p; s''_p}(x) = 0$ . So:

$$y_p(s''_p) = \frac{x_p(s''_p) + \text{Gain}_{p; s''_p}(x)}{1 + \sum_{s'_p \in S_p} \text{Gain}_{p; s'_p}(x)} < x_p(s''_p),$$

since the numerator is equal to  $x_p(s''_p)$ , while the denominator is greater than 1 as there is at least one non-zero gain in the summation—the one corresponding to pure strategy  $s_p$ . Therefore,  $x$  is not a fixed point, a contradiction. It follows that  $x$  is a Nash equilibrium, as desired.  $\square$

## References

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